08/02/2024 MPHYCC-3 Teaching: Quantum Physics: Online Mode

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[#] Poisson Bracket's And Commutators.

The Poisson Bracket of u and v w.r.to q_j, p_j is given by

$$\{u, v\} = \sum_i \left\{ \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} \right\} \quad (1)$$

Sly the Poisson brackets of q_j, p_k coordinates is given by

$$\{q_j, p_k\} = \sum_i \left(\frac{\partial q_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial p_k}{\partial q_i} \right)$$

$$= \delta_{jk} \quad (2)$$

δ_{jk} is Kronecker delta fn.

$\delta_{jk} = 1$ for $j = k$
$= 0$ for $j \neq k$

Thus $\{q_i, p_i\} = 1$

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$$\{u_1u_2, v_1v_2\} = u_1\{u_2, v_1v_2\} + \{u_1, v_1v_2\}u_2$$
$$= u_1\left(\{u_2, v_1\}v_2 + v_1\{u_2, v_2\}\right) + \left(\{u_1, v_1\}v_2 + v_1\{u_1, v_2\}\right)u_2$$

$$= u_1\{u_2, v_1\}v_2 + u_1v_1\{u_2, v_2\}$$
$$+ \{u_1, v_1\}v_2u_2 + v_1\{u_1, v_2\}u_2 \quad \text{(3)}$$

Sly, $\{u_1u_2, v_1v_2\}$
$$= \{u_1u_2, v_1\}v_2 + v_1\{u_1u_2, v_2\}$$

$$= u_1\{u_2, v_1\}v_2 + \{u_1, v_1\}u_2v_2$$
$$+ v_1u_1\{u_2, v_2\} + v_1\{u_1, v_2\}u_2 \quad \text{(4)}$$

Comparing expression (3) & (4), we get
$$u_1v_1\{u_2, v_2\} + \{u_1, v_1\}v_2u_2$$
$$= \{u_1, v_1\}u_2v_2 + v_1u_1\{v_2, u_2\}$$

or, $(u_1v_1 - v_1u_1)\{u_2, v_2\} = \{u_1, v_1\}(u_2v_2 - v_2u_2)$

As u_1 & v_1 are independent of u_2 and v_2 and therefore we have

$$\frac{u_1v_1 - v_1u_1}{[u_1, v_1]} = \frac{u_2v_2 - v_2u_2}{[u_2, v_2]} = \alpha \rightarrow \text{a constant} \rightarrow (6)$$

Thus $uv - vu = \alpha\{u, v\} \rightarrow (7)$

Now, if $u \equiv \hat{q} \rightarrow$ position operator

$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial q}$ then

$$(uv - vu)\psi = (\hat{q}\hat{p} - \hat{p}\hat{q})\psi$$

$$= \left(q\frac{\hbar}{i}\frac{\partial}{\partial q} - \frac{\hbar}{i}\frac{\partial}{\partial q}q\right)\psi$$

$$= q\frac{\hbar}{i}\frac{\partial\psi}{\partial q} - \frac{\hbar}{i}\left(q\frac{\partial\psi}{\partial q} + \psi\right)$$

$$= -\frac{\hbar}{i}\psi = i\hbar\psi$$

$\therefore (uv - vu) = qp - pq = i\hbar \rightarrow (8)$

& therefore the poisson bracket of $u = q$; $v = p$ i.e. $\{u, v\} = \{p, q\} = 1 \rightarrow (9)$

After using this relation (8) and (9), eqn (7) gives $\alpha = i\hbar \rightarrow (10)$

Eqn (7) takes the form

$(uv - vu) = i\hbar\{u, v\}$

$[u, v] = i\hbar\{u, v\} \rightarrow (11)$

$uu - vu = [u, v]$ = Commutator of u & v.

~~Eq~~ Thus it is obvious that the commutator & Poisson bracket of two operators differ only by a constant. Consequently all the properties of Poisson brackets must be obeyed by quantum mechanical commutators. Hence

$[q_i, q_j] = 0$ sin $\{q_i, q_j\} = 0$

$[p_i, p_j] = 0$ sin $\{p_i, p_j\} = 0$

Thus $[p_i, q_j] = i\hbar\delta_{ij}$

Commutator for Algebra

In Analogy with Poisson brackets, Quantum mechanical commutators have the following properties

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}] \quad \text{--- (i)}$$

$$[\hat{A}, \hat{B}+\hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] \quad \text{--- (ii)}$$

$$[\hat{A}+\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] \quad \text{--- (iii)}$$

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}] \quad \text{--- (iv)}$$

$$[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}] \quad \text{--- (v)}$$

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0 \quad \text{--- (vi)}$$

All the above relⁿ can be verified since

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

Commutation Relⁿ betⁿ Position & Momentum

Say $x \rightarrow \hat{x}$ & $p_x \rightarrow \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$$[\hat{x}, \hat{p}_x]\psi = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\psi$$

$$\hat{x}\hat{p}_x\psi = x\frac{\hbar}{i}\frac{\partial \psi}{\partial x}$$

$$\& \ \hat{p}_x\hat{x}\psi = \frac{\hbar}{i}\frac{\partial}{\partial x}(x\psi) = \frac{\hbar}{i}\left(\psi + x\frac{\partial \psi}{\partial x}\right) \quad \text{--- (1)}$$

$$\cancel{= \frac{\hbar}{i}\left(x\frac{\partial \psi}{\partial x}\right)}.$$

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Now $[\hat{x}, \hat{p}_x] = x\frac{\hbar}{i}\frac{\partial\psi}{\partial x} - \frac{\hbar}{i}\left(\psi + x\frac{\partial\psi}{\partial x}\right)$

$= \frac{\hbar}{i}\left(x\frac{\partial\psi}{\partial x} - \psi - x\frac{\partial\psi}{\partial x}\right)$

$= -\frac{\hbar}{i}\psi = i\hbar\psi$

$\Rightarrow [\hat{x}, \hat{p}_x] = i\hbar$ — (2)

Sly $[\hat{x}^2, \hat{p}_x] = [\hat{x}\hat{x}, \hat{p}_x]$

$= [\hat{x}, \hat{p}_x]\hat{x} + \hat{x}[\hat{x}, \hat{p}_x]$

Since $[ab, c] = [a, c]b + a[b, c]$

or $[\hat{x}^2, \hat{p}_x] = i\hbar\hat{x} + x i\hbar$

$= 2i\hbar\hat{x}$ — (3)

Sly $[\hat{x}^3, \hat{p}_x] = 3i\hbar\hat{x}^2$ — (4)

Sly by Induction we will have,

$\boxed{[\hat{x}^n, \hat{p}_x] = n i\hbar \hat{x}^{n-1}}$ — (5)

$\frac{\partial \psi}{\partial x} = \frac{\hbar}{i}\left(\psi + x\frac{\partial \psi}{\partial x}\right.$

$\left. \psi - x\frac{\partial \psi}{\partial x}\right)$

$= i\hbar\psi$

$= i\hbar \quad \text{---} (2)$

$= [\hat{x}\hat{x}, \hat{p}_x]$

$] \hat{x} + \hat{x}[\hat{x}, \hat{p}_x]$

$= [a, c]b + a[b, c]$

$= i\hbar\hat{x} + x i\hbar$

$= 2i\hbar\hat{x} \quad \text{---} (3)$

$\text{---} (4)$

$] = 3i\hbar\hat{x}^2$

duction we

ve,

$] = n i\hbar \hat{x}^{n-1} \text{---} (5)$

$\text{lly } [\hat{x}, \hat{p}_x^2] = [\hat{x}, \hat{p}_x]\hat{p}_x + \hat{p}_x[\hat{x}, \hat{p}_x]$

$= i\hbar\hat{p}_x + \hat{p}_x i\hbar$

$\therefore [\hat{x}, \hat{p}_x^2] = 2i\hbar\hat{p}_x$

$\text{lly } [\hat{x}, \hat{p}_x^3] = 3i\hbar\hat{p}_x^2$

Again by induction

$\boxed{[\hat{x}, \hat{p}_x^n] = n i\hbar\hat{p}_x^{n-1}} \quad (6)$

$$[\hat{x}, \hat{p}_x^2] = [\hat{x}, \hat{p}_x]\hat{p}_x + \hat{p}_x[\hat{x}, \hat{p}_x]$$

$$= i\hbar \hat{p}_x + \hat{p}_x i\hbar$$

$$\therefore [\hat{x}, \hat{p}_x^2] = 2i\hbar \hat{p}_x$$

Sly $[\hat{x}, \hat{p}_x^3] = 3i\hbar \hat{p}_x^2$

A gain by induction;

$$\boxed{[\hat{x}, \hat{p}_x^n] = n i\hbar \hat{p}_x^{n-1}}$$

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